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Algorithm for Optimal Chance Constrained Linear Assignment

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***Abstract*— In this paper, we design provably-good algorithms for task allocation in multi-robot systems in the presence of payoff uncertainty. We consider a group of robots that has to perform a given set of tasks where each robot performs at most one task. The payoffs of the robots doing the tasks are assumed to be Gaussian random variables with known mean and variances. The total payoff of the robots is a sum of the individual payoffs of all the robots. The goal is to find an assignment with maximum payoff that can be achieved with a specified probability irrespective of the realization of the random variable. This problem can be formulated as a chance constrained combinatorial optimization problem. We develop a novel deterministic technique to solve this chance constrained optimization problem that ensures that the chance constraints are always satisfied. Adopting the notion of risk-aversion from the economics literature, we formulate a risk-averse task allocation problem, which is a deterministic integer optimization problem. We prove that by repeatedly solving the risk-averse task allocation problem using a one-dimensional search on the risk aversion parameter we find a solution for the chance constrained optimization formulation of the linear assignment problem with uncertain payoffs. We provide simulation results on randomly generated data to demonstrate our approach and also compare our method to existing approaches.**

I. INTRODUCTION

Multi-robot task assignment is a fundamental problem that arises in a wide variety of application scenarios like manufacturing, automated transport of goods, environmental monitoring, and surveillance [1], [2]. Task allocation problems are mathematically modeled as combinatorial optimization problems. The basic version of the task allocation problem (also known as linear assignment problem in combinatorial optimization) is: *Given a set of agents (or robots) and a set of tasks, with each robot obtaining some payoff (or incurring some cost) for each task, find a one-to-one assignment of agents to tasks so that the overall payoff of all the agents is maximized (or cost incurred is minimized).* The basic task assignment problem can be solved (near) optimally in polynomial time by centralized algorithms [3], [4] and decentralized algorithms [5]. Much of the research in linear assignment problems has assumed known payoffs (exceptions include [6], [7]). In applications, the payoffs of the robots for tasks may not be known exactly and for deployment one may want to have algorithms for linear assignment with performance guarantees in spite of the uncertain payoffs. The goal of this paper is to develop algorithms with performance guarantees for linear assignment problems with uncertain payoffs.

We consider a set of robots $R = \{r_i\}$, $i = 1, \dots, n_r$, and a set of tasks $T = \{t_j\}$, $j = 1, \dots, n_t$. The payoff for a robot r_i for task t_j is a Gaussian random variable a_{ij}. Each robot can do at most one task[1]. In task allocation problems with uncertain payoffs, one would like to have an assignment with some guarantees on the quality of assignment (i.e., payoff achievable in the assignment) irrespective of the realization of the random payoffs. Technically, there are different objectives that can be used for the assignment problem depending on the model of uncertainty used for the payoffs. Two typical models of uncertainty are set theoretic models and probabilistic models. Set theoretic uncertainty models for linear assignment has been considered in [7]. In a set theoretic model of uncertainty, the uncertain payoffs are assumed to lie within some range of a minimum and maximum value and the objective is to maximize the worst-case payoff. In other words, the solution obtained maximizes the total payoff for the worst possible realization of the individual payoffs. Thus, this solution may be overly conservative when the realization of the worst case is a very low probability event.

For probabilistic models of uncertainty, two popular objectives are the expected payoff and the chance-constrained payoff [8]. Although the expected payoff maximization is a quite popular objective for decision making problems under uncertainty, the solution obtained is meaningful only in situations where the same problem has to be executed multiple times by the robot team and one is interested in the average performance of the team over the multiple scenarios. For any particular realization of payoffs the solution may be far off from the predicted solution. Consequently, the objective does not provide any guarantees for any particular scenario. Another objective is to have a probabilistic guarantee on the quality of assignment for any given scenario by using a chance constraint formulation, which is what we pursue in this paper. Our goal is to compute an assignment of robots to tasks such that: *Given a p, find an assignment that ensures that with probability p, we can obtain the maximum achievable payoff under any realization of the random variable*. When $p = 1$, we have a problem with worst-case guarantees. The above task allocation problem can be formulated as a chance-constrained combinatorial optimization problem.

Chance constrained optimization problems are a class of

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[1]Note that the problem where we have to minimize the cost of an assignment is identical and therefore we will talk about the payoff maximization problem here.

stochastic optimization problem [9], [10]. They are usually hard to solve (except for some special cases like linear optimization [9], minimum spanning tree [11]). In [12] the authors have presented algorithms for chance-constrained shortest path problems. In [13], the method of [12] has been extended to a class of chance constrained optimization problems, where the objective function is quasi-convex. The chance constrained linear assignment problem that we consider is a special case of the problem considered in [13]. The key aspect of [13] is to note that the chance constrained problem (also known as the value-at-risk formulation) is equivalent to the mean-risk formulation (where the objective function is a sum of the mean and a constant times the standard deviation). It is known [9] that the mean-risk problem is quasi-concave and the objective function depends on only the means and the variances of the feasible solutions. Thus, finding the optimal solution to the mean-risk problem reduces to finding the optimal solution to the mean-risk problem over the projection of the feasible points in the mean-variance plane. The authors then go on to provide an exact solution where all extreme points in the mean-variance plane are enumerated. This enumeration is actually done by solving a risk-averse problem for various values of the risk-aversion parameter, although the connection to the risk-averse problem formulation is implicit in the paper.

Our overall approach to solve the value-at-risk formulation of linear assignment builds on the work by [13]. We make the relationship between the risk averse problem and the value-at-risk problem explicit. Based on this insight, we obtain an upper bound on the risk-aversion parameter ($\tilde{\lambda}$) such that solving a finite number of deterministic risk-averse problems with values of risk-aversion parameter below $\tilde{\lambda}$ is guaranteed to provide us the optimal solution for the value-at-risk problem. We present a methodical one-dimensional search method on the risk-aversion parameter and show through simulations on random data sets that our method is much more efficient than the exact method proposed in [13]. The connection between the value-at-risk problem and the risk-averse problem for linear assignment as well as the deterministic method to solve the value-at-risk problem that we present here are the primary contributions of this paper.

A. *Other Related Work*

Task allocation is important in many applications of multi-robot systems, e.g., multi-robot routing [14], multi-robot decision making [15], and other multi-robot coordination problems (see [2], [16]). There are different variations of the multi-robot assignment problem that have been studied in the literature depending on the assumptions about the tasks and the robots (see [1], [2], [17] for surveys), and there also exists multi-robot task allocation systems (e.g., Traderbot [18], [19], Hoplites [20], MURDOCH [21], ALLIANCE [22]) that build on different algorithms. However, chance constrained problems have been studied in multi-robot task allocation in only a handful of papers.

In [8], [23] the authors model the multi-robot routing problem as a chance constrained optimization problem. The payoffs are also time-sensitive in that the payoff of visiting a target site reduces with time. Hence all tasks need not be done by the robots. The relationship between the chance constrained and risk-averse problem has not been identified or explored in [8], [23]. In [24], the authors have extended the work in [12] to develop faster algorithms for stochastic shortest path problems.

II. Formulations of Linear Assignment Problem Under Payoff Uncertainty

In this section we will describe the chance constrained optimization formulation for the linear assignment problem (LAP). We will first present the basic formulation of the linear assignment problem and then introduce the chance constrained version. Suppose that there are n_r robots, $R = \{r_1, \dots, r_{n_r}\}$, and there are n_t tasks, $T = \{t_1, \dots, t_{n_t}\}$. Let the payoff of robot r_i for performing tasks t_j be a_{ij}. Let $J = \{j_1, j_2, \dots, j_{n_r}\}$ be the index set of the tasks in which each element is the index of task assigned to a robot. For example, if robot r_i is assigned to tasks t_u, then $j_i = u$. We assume that any robot can be assigned to any task. If a robot cannot be assigned to a task we can model it by making the corresponding payoff as $-\infty$. Furthermore, we assume that performing each task needs a single robot and the number of tasks is same with the number of robots, i.e., $n_r = n_t = n$. In LAP, the goal is to assign robots to tasks such that the overall robot team payoff is maximized. Let f_{ij} be a binary variable such that $f_{ij} = 1$ if robot r_i is assigned to task t_j and 0 otherwise. The integer programming (IP) formulation of LAP is

$$\begin{aligned} &\max \sum_{i=1}^{n_r} \sum_{j=1}^{n_t} a_{ij} f_{ij} \\ &\quad \text{s.t.} \sum_{i=1}^{n_r} f_{ij} = 1, \quad \forall j, \quad \sum_{j=1}^{n_t} f_{ij} = 1, \quad \forall i, \quad f_{ij} \in \{0,1\} \end{aligned} \tag{1}$$

The first set of constraints imply that each task can be done by exactly one robot. The second set of constraints encode the fact that each robot can perform exactly one task. We will call the constraints in the LAP as the *linear assignment constraints*.

This is a typical LAP problem which is a classical problem in operations research and combinatorial optimization and there are many polynomial time algorithms that can solve LAP optimally. In LAP it is usually assumed that the task payoffs, a_{ij}, are known. If the task payoffs are not known exactly but a distribution of the task payoffs is known, then the problem formulation in Equation (1) (more precisely, the objective function) has to be modified. The objective function can be (a) expected value of the total payoff over the random variables (b) robust total payoff (c) chance constrained payoff and (d) risk-averse payoff. We discuss the chance constrained and risk averse formulations below.

A. *Chance Constrained Linear Assignment Problem*

The chance constrained linear assignment problem (CC-LAP) allows one to obtain a solution with a probabilistic guarantee in situations where the payoffs are uncertain. The CC-LAP can be formulated as an integer program with probabilistic (or chance) constraints.

$$\begin{aligned} \max \quad & y \\ \text{s.t.} \quad & \mathbb{P}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_t} a_{ij} f_{ij} > y\right) \geq p \\ & \sum_{i=1}^{n_r} f_{ij} = 1, \quad \forall j, \quad \sum_{j=1}^{n_t} f_{ij} = 1, \quad \forall i, \quad f_{ij} \in \{0,1\} \end{aligned} \tag{2}$$

where p is a given parameter. The first constraint ensures that irrespective of the realization of the random variable, a_{ij}, the total payoff obtained is always greater than y with a probability of p with $0.5 \leq p \leq 1$. This constraint is the chance constraint of the optimization problem. For example, if we choose $p = 0.95$, it implies that the solution to Equation (2) gives a task assignment with the maximum payoff y that ensures that no matter what, the overall payoff will be always greater than y in 95% of the scenarios. The chance constrained problem formulation is also known as the value-at-risk problem formulation. A feasible solution to the CC-LAP has to satisfy both the chance constraints and the linear assignment constraints.

Solving a chance constrained integer optimization problem is in general quite hard and usually Monte Carlo simulations are used to satisfy the chance constraint. Another method to formulate and solve decision making problems under uncertainty is to use the notion of a risk-averse solution. We will present the risk-averse task allocation problem formulation for LAP in the next section and show the connection between the risk-averse problem and the chance constrained problem. Then we will present a deterministic algorithm that will allow us to solve the chance constrained problem by solving multiple instances of the risk-averse optimization problem. Note that the risk-averse optimization formulation can be interesting in its own right. Although RA-LAP cannot give *a priori* guarantee on the quality of the solution it can give *a posteriori* guarantee. In other words, given a value of the risk aversion parameter, after solving the problem we can determine the probability that the obtained solution will be the best solution for any realization of the payoffs.

B. Risk Averse Linear Assignment Problem

Assume that the payoffs of a robot, r_i for the tasks, t_j are drawn from a joint Gaussian distribution. Let μ_{ij} be the expected payoff for robot r_i performing task t_j, i.e., $E[a_{ij}] = \mu_{ij}$. Similarly, let σ_{ij}^2 be the variance of the payoff, i.e., $Var(a_{ij}) = \sigma_{ij}^2$.

For any assignment that satisfies the linear assignment constraints, let $j_i \in J$ be the index of the task assigned to robot r_i. The payoff for robot i, denoted by P_i is then a_{ij_i} and the total payoff of the robot team is

$$P = \sum_{i=1}^{n_r} P_i = \sum_{i=1}^{n_r} a_{ij_i} = \sum_{i=1}^{n_r}\sum_{j=1}^{n_t} a_{ij} \tag{3}$$

We define a utility function, U, for the robot team as

$$U = -e^{-\lambda P} = -e^{-\lambda \sum_{i=1}^{n_r} a_{ij_i}} \tag{4}$$

where $\lambda > 0$ is the index of risk-aversion for the robot team (in economics, this is known as the Arrow-Pratt index of absolute risk aversion). The higher the value of λ the more risk averse the robot. The utility function U has the following properties: (1) It is non-positive with its value equal to zero for any feasible assignment if $\lambda = 0$. (2) For any given λ, the utility U is a monotonically increasing function of the total payoff P. Since a_{ij} is a stochastic variable, instead of maximizing the total payoff of assignment, one may want to maximize the total expected utility of the assignment. Note that the total payoff P is a Gaussian random variable since $P_i = a_{ij_i}$ is a Gaussian random variable. Furthermore, for any feasible assignment, two robots will not be assigned to the same task. Thus P_i are statistically independent. Therefore, for any feasible assignment, the total payoff P is also a Gaussian random variable with mean $\mu = \sum_{i=1}^{n_r} \mu_{ij_i}$ and variance $\sigma^2 = \sum_{i=1}^{n_r} \sigma_{ij_i}^2$. Therefore, from [25]

$$E[U] = -e^{-2\lambda(\mu - \lambda\sigma^2)} \tag{5}$$

Thus, for any given λ, maximizing $E[U]$ is equivalent to maximizing $\mu - \lambda\sigma^2$. Now,

$$\mu - \lambda\sigma^2 = \sum_{i=1}^{n_r}\sum_{j=1}^{n_t} (\mu_{ij} - \lambda\sigma_{ij}^2) f_{ij} \tag{6}$$

Thus we can formulate the risk-averse optimization formulation for the linear assignment problem as follows:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n_r}\sum_{j=1}^{n_t} (\mu_{ij} - \lambda\sigma_{ij}^2) f_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n_r} f_{ij} = 1, \quad \forall j, \quad \sum_{j=1}^{n_t} f_{ij} = 1, \quad \forall i, \quad f_{ij} \in \{0,1\} \end{aligned} \tag{7}$$

where f_{ij} are the optimization variables. By solving the risk-averse task assignment problem (RA-LAP) with a given value of λ, we can obtain the optimal assignment $J(\lambda)$ and optimal objective function value $Q(\lambda) = \sum_{i=1}^{n_r} \mu_{ij_i} - \lambda \sum_{i=1}^{n_r} \sigma_{ij_i}^2$. Further the mean and variance of total payoff with the assignment $J(\lambda)$ can be computed, i.e., $\mu = \sum_{i=1}^{n_r} \mu_{ij_i}, \sigma^2 = \sum_{i=1}^{n_r} \sigma_{ij_i}^2$. We will use the notations above extensively in this paper.

The objective function in Equation (7) is also known as the mean-variance payoff function for the robot team. The mean-variance payoff function encodes the following:

1) The variance of the payoff distribution of an assignment is a measure of risk for the assignment.
2) If there are two assignments with the same expected payoff, the assignment with the lower variance is preferred.
3) If there are two assignments with the same payoff variance, the assignment with the higher expected payoff is preferred.

The two problem formulations that we have introduced for taking into consideration stochasticity has complementary strengths. The solution to the chance constrained problem formulation ensures an *a priori* probabilistic guarantee of the solution quality. However, it is difficult to solve. On the other hand the risk averse formulation is easy to solve but provides only an *a posteriori* guarantee on the solution quality depending on the value of the risk aversion parameter λ. Therefore, in the next section we will discuss a relationship between the solutions of the chance constrained and the risk averse problem that will allow us to develop an algorithm for solving the chance constrained problem by repeatedly solving the risk-averse problems.

C. Connection between Chance Constrained and Risk-Averse Formulations

In this section, we argue that (a) the solution of RA-LAP for any value of λ can always give a feasible solution to the chance constrained problem and (b) the optimal solution of the CC-LAP is the optimal solution of the RA-LAP with some value of λ. Therefore we can solve CC-LAP by solving RA-LAP.

Consider the chance constraint in CC-LAP. For any assignment that satisfies the linear assignment constraints, the total payoff, $P = \sum_{i=1}^{n_r}\sum_{j}^{n_t} a_{ij}f_{ij}$, is a Gaussain random variable with mean μ and variance σ^2. Therefore, the chance constraint of CC-LAP can be equivalently written as:

$$\mathbb{P}(P > y) = 1 - \frac{1}{2}\left(1 + \text{erf}\left(\frac{y-\mu}{\sqrt{2}\sigma}\right)\right) \tag{8}$$

where $\text{erf}(\cdot)$ is the error function. Let $g(y) = \mathbb{P}(P > y)$.

$$\therefore\ g(y) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{y-\mu}{\sqrt{2}\sigma}\right)\right) \geqslant p \tag{9}$$

$$\Rightarrow y \leqslant \mu - [\sqrt{2}\text{erf}^{-1}(2p-1)]\sigma \tag{10}$$

Let $C = \sqrt{2}\text{erf}^{-1}(2p-1)$, which is a constant when the required probability p is given. Therefore, any assignment that satisfies the linear assignment constraints is also a feasible solution to CC-LAP with maximum value of $y = \mu - C\sigma$. Thus solving CC-LAP is equivalent to finding the assignment (that satisfies the linear assignment constraints) with maximum value of $\mu - C\sigma$. The optimal solution to RA-LAP for any value of the risk aversion parameter λ satisfies the linear assignment constraints. Therefore, it is true that any optimal assignment for the RA-LAP is feasible for the CC-LAP where the objective $y = \mu - C\sigma$, where μ and σ^2 are the mean and variance of the total payoff of the optimal assignment.

We now argue that the optimal assignment for the CC-LAP is an optimal assignment to the RA-LAP for some value of the risk-averse parameter λ. Note that each feasible assignment i to CC-LAP (Equation (2)) has a mean μ_i, variance σ_i^2, and a y_i. Thus with each assignment, we can associate a point in a two dimensional plane with coordinates (σ_i^2, μ_i). Figure 1 shows a schematic sketch of the variance-mean plane with circles indicating the different feasible assignments. We first note that the optimal solution to the CC-LAP will be an extreme point of this point set in the σ^2-μ plane [12], [26]. Furthermore, the objective function of the RA-LAP is a line in the σ^2-μ plane with the risk-aversion parameter, λ, as the slope. Therefore, the optimal solution for the RA-LAP for any value of λ is an extreme point of the point set. Thus, we can conclude that the optimal solution for CC-LAP will be the optimal solution for RA-LAP for some value of λ.

The above two facts motivate our solution approach of performing a one-dimensional search on the risk aversion parameter, λ, to find the risk aversion parameter (say λ^*) corresponding to the optimal solution of CC-LAP. Our Algorithm 1 bounds the possible values of λ^*, i.e., $\lambda^* \in [0, \tilde{\lambda}]$. Notice that $\tilde{\lambda}$ gives a lower bound of the optimal objective value of CC-LAP. This fact is discussed in Section III. The Algorithm 2 searches for all the extreme points by solving RA-LAP with λ within this range. Compared with the method in [12], where the range of λ is $[0, \infty]$, we limit the search set for λ, which makes our algorithm more efficient (please see Section IV for the computational comparisons).

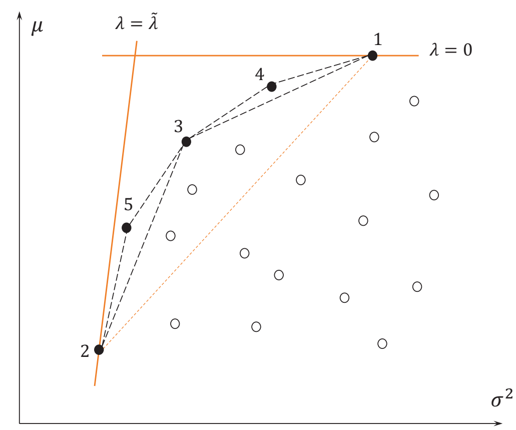

Fig. 1. Projection of feasible assignments on σ^2-μ plane

III. Algorithm for Solving Task Allocation with Chance Constraints

In this section we present our algorithm for solving the chance constrained optimization problem. Before presenting our algorithm, we first provide a few basic facts about the solution of RA-LAP and establish the relationship between the CC-LAP and the RA-LAP.

Lemma 1: The optimal objective function value for RA-LAP is a strictly monotonically decreasing function of the risk aversion parameter, i.e., $\mu_1 - \lambda_1\sigma_1^2 > \mu_2 - \lambda_2\sigma_2^2$ for any $\lambda_2 > \lambda_1$, where μ_1 and σ_1^2 are the mean and variance of the total payoff with the optimal assignment obtained from

RA-LAP with λ_1. The same way of notation is applied for μ_2 and σ_2^2.

Proof: Let λ_1 and λ_2 be two different risk aversion parameters, and $\lambda_1 < \lambda_2$. μ_1 and σ_1^2 are the mean and variance of total payoff that maximize the objective function of RA-LAP $\sum_{i=1}^{n_r}\sum_{j=1}^{n_t}(\mu_{ij} - \lambda\sigma_{ij}^2)f_{ij}$. Therefore

$$\mu_1 - \lambda_1\sigma_1^2 > \mu_2 - \lambda_1\sigma_2^2$$

Since $\lambda_2 > \lambda_1$

$$\mu_2 - \lambda_1\sigma_2^2 > \mu_2 - \lambda_2\sigma_2^2$$

From inequalities above, we conclude

$$\mu_1 - \lambda_1\sigma_1^2 > \mu_2 - \lambda_2\sigma_2^2$$

■

Lemma 2: The variance of total payoff with the assignment obtained from RA-LAP is a monotonically decreasing function of risk aversion parameter, i.e., $\sigma_1^2 > \sigma_2^2$ for any $\lambda_1 < \lambda_2$.

Proof: Since μ_2 and σ_2^2 are optimal for the RA-LAP with λ_2, it is true that

$$\mu_1 - \lambda_2\sigma_1^2 < \mu_2 - \lambda_2\sigma_2^2 \tag{11}$$

Similarly,

$$\mu_1 - \lambda_1\sigma_1^2 > \mu_2 - \lambda_1\sigma_2^2 \tag{12}$$

Subtracting Equation (12) from Equation (11), we obtain

$$(\lambda_1 - \lambda_2)\sigma_1^2 < (\lambda_1 - \lambda_2)\sigma_2^2$$

Since $\lambda_1 < \lambda_2$, we have

$$\sigma_1^2 > \sigma_2^2$$

■

Lemma 3: There exists a risk aversion parameter, denoted by $\tilde{\lambda}$, such that the optimal objective function value of RA-LAP with $\tilde{\lambda}$ is equal to the objective value of CC-LAP, i.e., $y = \tilde{\mu} - C\tilde{\sigma} = \tilde{\mu} - \tilde{\lambda}\tilde{\sigma}^2$. This objective value gives a lower bound for the optimal objective value of CC-LAP. Furthermore, the value of the risk-aversion parameter, λ^*, that gives the optimal assignment for CC-LAP must lie in the interval $[0, \tilde{\lambda}$.

Proof: First we need to prove that $\tilde{\lambda}$ exists. As is mentioned in Lemma 1, $\mu - \lambda\sigma^2$ is a monotonically decreasing function of λ. It is possible that the optimal assignment of RA-LAP remains same even though λ changes. But $\mu - \lambda\sigma^2$ still decreases. Therefore the optimal objective function value of RA-LAP will always decrease as long as λ increases. For CC-LAP, the objective value $\mu - C\sigma$ changes only when the optimal assignment of RA-LAP changes. Since the feasible assignment of RA-LAP is finite, there exists one risk aversion parameter, denoted by λ_m, such that the optimal assignment of RA-LAP with $\lambda > \lambda_m$ is same with the assignment of RA-LAP with λ_m. Then $\mu - C\sigma$ remains the same while $\mu - \lambda\sigma^2$ keeps decreasing. Therefore the risk aversion parameter that satisfies the equation $\mu - C\sigma = \mu - \lambda\sigma^2$ exists. Notice that there are possibly multiple values for such λ, but our algorithm provides the smallest one. We will justify this claim in the other part of this paper.

Now we need to prove that the objective values of CC-LAP obtained outside the range $[0, \tilde{\lambda}]$ are smaller than the objective value obtained at $\tilde{\lambda}$, i.e., $\tilde{\mu} - C\tilde{\sigma} > \mu - C\sigma$ for any $\lambda > \tilde{\lambda}$. If this is true, then we can conclude that $\tilde{\lambda}$ gives a lower bound of optimal objective value of CC-LAP and the risk aversion parameter λ^* that gives the optimal assignment must lie in the range $[0, \tilde{\lambda}]$. Suppose μ and σ^2 are obtained by solving RA-LAP with $\lambda > \tilde{\lambda}$, there are two different cases:

- $\lambda\sigma \leqslant C$. Obviously $y = \mu - C\sigma \leqslant \mu - \lambda\sigma^2$. And because $\lambda > \tilde{\lambda}$, from Lemma 1 we know that $\mu - \lambda\sigma^2 < \tilde{\mu} - \tilde{\lambda}\tilde{\sigma}^2 = \tilde{\mu} - C\tilde{\sigma}$. Therefore $\tilde{\mu} - C\tilde{\sigma} > \mu - C\sigma$.
- $\lambda\sigma > C$. Let $\lambda' = C/\sigma$ and $\lambda' < \lambda$. Since $\lambda > \tilde{\lambda}$, from Lemma 2 we know $\sigma < \tilde{\sigma}$ and further $\lambda' > \tilde{\lambda}$. Therefore $\mu - C\sigma = \mu - \lambda'\sigma^2 < \mu' - \lambda'\sigma'^2 < \tilde{\mu} - \tilde{\lambda}\tilde{\sigma}^2 = \tilde{\mu} - C\tilde{\sigma}$.

The analysis shows that the $\tilde{\mu} - C\tilde{\sigma}$ is the lower bound of optimal objective value for CC-LAP. And since the risk aversion parameter is a positive number, λ^* that gives the optimal assignment must be in range $[0, \tilde{\lambda}]$. ■

From the lemma above, we design the Algorithm 1 that can efficiently find $\tilde{\lambda}$. The inputs of this algorithm are the required probability p, mean μ_{ij} and variance σ_{ij}^2 of each task. The outputs are the set of risk aversion parameters which includes both 0 and $\tilde{\lambda}$ that gives the lower bound of optimal objective value of CC-LAP and the assignment set includes the optimal assignment of RA-LAP with risk aversion parameter 0 and $\tilde{\lambda}$. Line 1 computes the constant C related to chance constraint. Line 2 solves the RA-LAP with $\lambda = 0$. The while loop of line 3-6 computes the risk aversion parameter, solves RA-LAP with this parameter and checks whether $\lambda\sigma = C$. If $\lambda\sigma \neq C$, line 4 computes the risk aversion parameter in the next while iteration. The while loop keeps running unless the condition is true. Line 7 forms the risk aversion parameter set and assignment set. The Algorithm 1 allows us to have following claim.

Claim 1: Algorithm 1 finds the smallest risk aversion parameter λ such that $\mu - C\sigma = \mu - \lambda\sigma^2$.

Proof: Suppose the risk aversion parameter is $\lambda_i \in [0, \tilde{\lambda}]$, μ_i and σ_i^2 are obtained from RA-LAP with λ_i. If $\lambda_i\sigma_i < C$, the risk aversion parameter in the next step can be obtained by the method introduced in Algorithm 1, i.e., $\lambda_{i+1} = \frac{C}{\sigma_i}$. It is true for any $\lambda \in [\lambda_i, \lambda_{i+1})$ that

$$\lambda\sigma < \lambda_{i+1}\sigma \leqslant \lambda_{i+1}\sigma_i = C$$

Also $\sigma_{i+1} < \sigma_i$, therefore

$$\lambda_{i+1}\sigma_{i+1} < \lambda_{i+1}\sigma_i = C$$

Algorithm 1 essentially proceeds in a way that i change from 0 to some value of k ($\lambda_k = \tilde{\lambda}$). This implies that for any $\lambda \in [0, \tilde{\lambda})$, $\mu - \lambda\sigma^2 > \mu - C\sigma$. So we can conclude that Algorithm 1 finds the smallest λ such that $\lambda\sigma = C$. ■

After finding $\tilde{\lambda}$ which gives the lower bound of optimal objective value of CC-LAP, we need to search for the optimal assignment in the range $[0, \tilde{\lambda}]$. More precisely we need to search for extreme points on σ^2-μ plane, as is shown in Figure 1.

Lemma 4: For two different risk aversion parameters, $\lambda_i > \lambda_j$ and the corresponding RA-LAP have different optimal assignments $J(\lambda_i) \neq J(\lambda_j)$, there exists a risk aversion parameter $\bar{\lambda} \in (\lambda_i, \lambda_j)$ such that if $J(\bar{\lambda}) = J(\lambda_i)$ or $J(\bar{\lambda}) = J(\lambda_j)$, then the optimal assignment for RA-LAP with $\lambda \in (\lambda_i, \bar{\lambda})$ is same with the optimal assignment at λ_i, i.e., $J(\lambda) = J(\lambda_i)$ and similarly $J(\lambda) = J(\lambda_j)$ for any $\lambda \in (\bar{\lambda}, \lambda_j)$.

Proof: Since $\mu_i - \lambda_i\sigma_i^2, \mu_j - \lambda_j\sigma_j^2$ are optimal for RA-LAP at λ_i, λ_j respectively, we have $\mu_i - \lambda_i\sigma_i^2 > \mu_j - \lambda_i\sigma_j^2$ and $\mu_i - \lambda_j\sigma_i^2 < \mu_j - \lambda_j\sigma_j^2$. Consider $\mu_i, \mu_j, \sigma_i^2, \sigma_j^2$ as constants, then $f(\lambda) = \mu_i - \lambda\sigma_i^2$ and $s(\lambda) = \mu_j - \lambda\sigma_j^2$ are linear functions. Imagine $f(\lambda)$ and $s(\lambda)$ as two line segments between λ_i and λ_j. Because $f(\lambda_i) > s(\lambda_j)$ and $f(\lambda_j) < s(\lambda_j)$, there must exist an intersection between λ_i and λ_j, where $f(\lambda) = s(\lambda)$. The value of λ at the intersection is defined as $\bar{\lambda}$ which is equal to $\frac{\mu_i - \mu_j}{\sigma_i^2 - \sigma_j^2}$.

Now we define another function $h(\lambda) = \mu_k - \lambda\sigma_k^2$, where μ_i, σ_k^2 are obtained from any feasible assignment of RA-LAP with $\lambda \in (\lambda_i, \lambda_j)$ and we consider μ_k, σ_k^2 as constants in this function. Therefore $h(\lambda)$ is a linear function. Because $\mu_i - \lambda_i\sigma_i^2, \bar{\mu} - \bar{\lambda}\bar{\sigma}^2$ are optimal for RA-LAP at $\lambda_i, \bar{\lambda}$ respectively, $h(\lambda_i) < f(\lambda_i)$ and $h(\bar{\lambda}) < \bar{\mu} - \bar{\lambda}\bar{\sigma}^2$. If $J(\bar{\lambda}) = J(\lambda_i)$, then $f(\bar{\lambda}) = \mu_i - \bar{\lambda}\sigma_i^2 = \bar{\mu} - \bar{\lambda}\bar{\sigma}^2$. Therefore $f(\bar{\lambda}) > h(\bar{\lambda})$. Further since both $f(\lambda)$ and $h(\lambda)$ are linear, $f(\lambda) > h(\lambda)$ for any $\lambda \in (\lambda_i, \bar{\lambda})$. This is true for any feasible solution of RA-LAP with $\lambda \in (\lambda_i, \bar{\lambda})$. This implies that $J(\lambda_i)$ is the optimal assignment of RA-LAP over the range $(\lambda_i, \bar{\lambda})$. It similar to prove that the $J(\lambda_j)$ is the optimal assignment of RA-LAP over the range $(\bar{\lambda}, \lambda_j)$. We can further conclude that $\bar{\lambda}$ is the risk aversion parameter where the optimal assignment of RA-LAP changes. In other words, $(\bar{\sigma}^2, \bar{\mu})$ is an extreme point on σ^2-μ plane of Figure 1. ■

Using Lemma 4, we design Algorithm 2, which searches for the optimal assignment of CC-LAP within the range obtained by Algorithm 1. The inputs of this algorithm are the range of risk aversion parameter $\lambda = \{0, \tilde{\lambda}\}$ and the corresponding assignments $J = \{J(0), J(\tilde{\lambda})\}$. The outputs are optimal assignment of CC-LAP J^* and optimal objective value y^*. Line 2 compute the current number of intervals, denoted by l. The for loop of line 3-12 does the same procedure discussed in Lemma 4 for l number of intervals. Line 4 computes $\bar{\lambda}_k$ for each interval. Line 5 solves RA-LAP with $\bar{\lambda}_k$ and then obtains the optimal assignment $\bar{J}_k$ and objective value $\bar{y}_k$ of CC-LAP with this assignment. Line 6 verifies whether $\bar{J}_k$ is same with the assignments at two endpoints of current interval, i.e., $J_{k,i}, J_{k,j}$. If it is true, that means we find an extreme point at $\bar{\lambda}_k$. Since there is no other extreme points in this interval, line 8 removes risk aversion parameters and the corresponding assignments from the set that is used for the next step. If it is false, there might be other extreme points between the current interval. So line 10 put the risk aversion parameter $\bar{\lambda}_k$, assignment $J(\bar{\lambda}_k)$ and objective value $\bar{y}_k$ into the active set. The for loop does the computation in this way for all intervals in the set. For the interval that $J(\bar{\lambda})$ is different with two assignment at endpoints, there are two small intervals generated from it in the next while iteration, i.e., $[\lambda_{k,i}, \bar{\lambda}_k]$ and $[\bar{\lambda}_k, \lambda_{k,j}]$. The while loop keeps running until the set of active assignment J is empty. Line 14 finds the optimal objective value y^* of CC-LAP and the optimal assignment J^*.

Figure 1 illustrates our procedure. Point 1 and point 2 are the projection of optimal assignments of RA-LAP at λ_1 and λ_2 on σ^2-μ plane. Then we can compute $\bar{\lambda} = \frac{\mu_1 - \mu_2}{\sigma_1^2 - \sigma_1^2}$, which is the slope of line passing through point 1 and point 2. Next the RA-LAP with $\bar{\lambda}$ is solved and the projection of the $J(\bar{\lambda})$ is point 3. If $J(\lambda)$ is different with assignment at point 1 and point 2, then we compute the slope of line connecting point 1 and point 3 and the slope of line connecting point 3 and point 2, i.e., $\bar{\lambda}_1 = \frac{\mu_1 - \bar{\mu}}{\sigma_1^2 - \bar{\sigma}^2}$ and $\bar{\lambda}_2 = \frac{\bar{\mu} - \mu_2}{\bar{\sigma}^2 - \sigma_2^2}$. Then we solve the RA-LAP with $\bar{\lambda}_1$ and $\bar{\lambda}_2$. If the assignment is same with assignment at any of two endpoints of the interval, e.g., $J(\bar{\lambda}) = J(\lambda_1)$, then the projection of $J(\bar{\lambda})$ has the same position with $J(\lambda_1)$ on the plane. Therefore point 1 is an extreme point and there is no other extreme point between point 1 and point 2. Then Algorithm 2 stops searching for optimal assignment between λ_1 and λ_2. Finally all extreme points in the triangle with red edges would be found.

Algorithm 1 Find the range

Input: $p, \mu_{ij}, \sigma_{ij} \forall i, j = 1, 2, \ldots n.$

Output: $\lambda, J.$

Compute $C = \sqrt{2}\text{erf}^{-1}(2p - 1)$

$\lambda = 0$, solve risk-averse problem, obtain the assignment J_0 and μ_0, σ_0

while $\lambda\sigma \neq C$ **do**

  compute $\lambda = \frac{C}{\sigma}$

  solve RA-LAP with λ, obtain J, μ, σ

end while

$\tilde{\lambda} = \lambda$ and $\tilde{J} = J$

$J = \{J_0, \tilde{J}\}, \lambda = \{0, \tilde{\lambda}\}$

IV. SIMULATION RESULTS

In Section III, we provided algorithms for computing a lower bound of the CC-LAP (Algorithm 1) and computing the optimal solution of CC-LAP (Algorithm 2). Since the CC-LAP is an instance of the class of problems considered in [13], the exact algorithm in [13] can also be used to solve the CC-LAP. The common aspect of all of the above algorithms is that it solves multiple instances of the RA-LAP to compute a solution for the CC-LAP. Therefore, a good measure of the efficiency of the algorithms is the number of RA-LAP problems that are solved. For the lower bound, we also need to compute the closeness of the solution to the optimal solution. In this section we use the above two metrics to compare the performance of the solution obtained using $\lambda = \tilde{\lambda}$ from Algorithm 1, $\lambda = \lambda^*$ from Algorithm 2 and the solution of [13]. We use randomly generated data sets to compare the different algorithms.

Algorithm 2 Search for the optimal solution within the interval

Input: λ, J

Output: J^*, y^*.

while $J \neq \emptyset$ and $\lambda \neq \emptyset$ **do**
  Compute the number of intervals s
  for $k = 1 : l$ **do**
    compute $\bar{\lambda}_k = \frac{\mu_{k,i} - \mu_{k,j}}{\sigma^2_{k,i} - \sigma^2_{k,j}}$
    solve RA-LAP, obtain $\bar{J}_k$ and $\bar{y}_k$
    if $\bar{J}_k \in \{J_{k,i}, J_{k,j}\}$ **then**
      Remove interval k
      $J = J \setminus \{J_{k,i}, J_{k,j}\}, \lambda = \lambda \setminus \{\lambda_{k,i}, \lambda_{k,j}\}$
    else
      $J = J \cup \bar{J}_k, \lambda = \lambda \cup \bar{\lambda}_k, Y = Y \cup \bar{y}_k$
    end if
  end for
end while
$y^* = \{y | y > y', \forall y' \in Y\}$ and J^* is corresponding assignment.

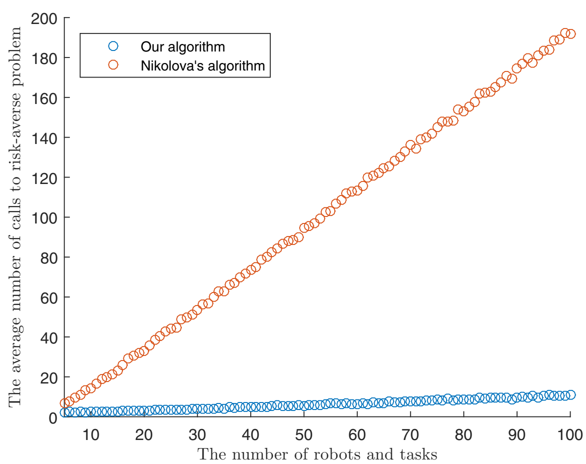

Fig. 2. Comparison between our algorithm and Nikolova's algorithm regarding the number of calls to risk-averse problem

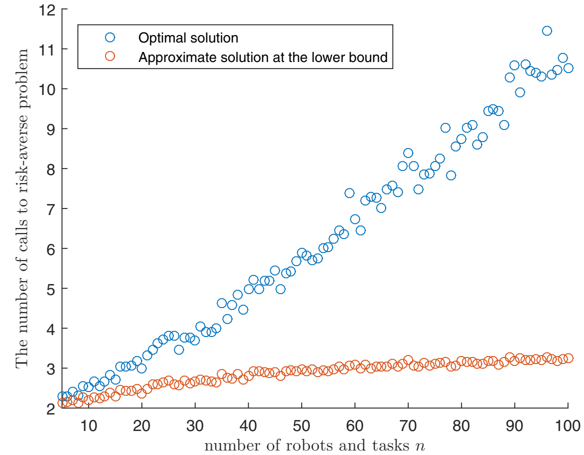

Fig. 3. Comparison between our algorithm that outputs exact solution and the algorithm that merely outputs approximate solution regarding the number of calls to risk-averse problem

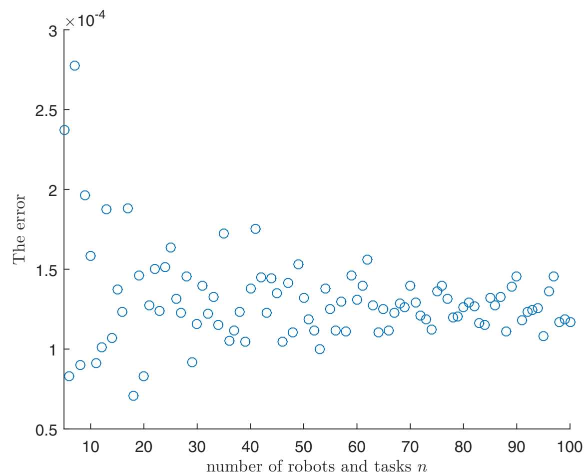

Fig. 4. The error of approximate solution

Recall that the payoff of robot r_i for performing task t_j is a_{ij}, which is a Gaussian random variable, i.e., $a_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma^2_{ij})$. For each problem, the mean and variance of payoffs are generated according to a uniform distribution in $(0, 100)$ and $(0, 20)$ respectively. We use our algorithm to solve a large number of stochastic linear assignment problems with increasing size from 5 to 100 with increment of 1. The problem with a given number of robots/tasks is solved for 100 times and the average number of calls to risk-averse problem is computed. We also use the method from [13] to do the same procedure. Figure 2 gives the comparison between our procedure for computing the optimal solution and the method from [13]. The number of calls to the risk-averse problem increases for both algorithms linearly with the number of robots/tasks. However, the rate of growth for our algorithm is much smaller compared to [13]. For example, for 100 robots, our algorithm solves about 11 RA-LAP problems, whereas the algorithm in [13] solves about 190 RA-LAP.

Figure 3 and 4 compares the number of calls to RA-LAP and the difference between the objective values for our lower bound and the optimal solution. As we can see from Figure 3, for computing the lower bound, the number of calls to RA-LAP is relatively constant (about 3 on average), whereas the number of calls to RA-LAP, although small, increases linearly. Furthermore, from Figure 4 the percentage difference between the two solutions is very small (of the order of $1e{-}04$). This seems to suggest that in practical situations, the lower bound that we are obtaining may be good enough. The reason of the good approximation is currently unclear and a subject of ongoing investigations. In fact, the question of whether the lower bound that we compute has any theoretical approximation guarantee is unknown.

V. Conclusion

In this paper, we presented provably-good algorithms for task allocation in multi-robot systems with payoff uncertainty. We formulated a chance constrained linear assignment problem and developed a novel deterministic technique to solve this chance constrained problem. Adopting the notion of risk-aversion from the economics literature, we formulate a risk-averse task allocation problem. We prove that by repeatedly solving the risk-averse task allocation problem using a one-dimensional search on the risk aversion parameter we find a solution for the chance constrained optimization problem. We provide simulation results on randomly generated data to demonstrate our approach and also compare our method to existing approaches. Although we demonstrated empirically that our algorithm is quite efficient, in the future, we would like to have a theoretical analysis of the computational complexity of our algorithm. In particular, we want to obtain a theoretical bound on the number of calls to the risk-averse problem for our algorithm. Furthermore, we want to explore the application of the relationship between the value-at-risk and risk-averse problem, to other variations of task allocation problems.

References

[1] B. P. Gerkey and M. J. Mataric, "A formal analysis and taxonomy of task allocation in multi-robot systems," *International Journal of Robotics Research*, vol. 23, no. 9, pp. 939–954, 2004.

[2] M. Dias, R. Zlot, N. Kalra, and A. Stentz, "Market-based multirobot coordination: A survey and analysis," *Proceedings of the IEEE*, vol. 94, no. 7, pp. 1257 –1270, jul. 2006.

[3] H. W. Kuhn, "The Hungarian method for the assignment problem," *Naval Research Logistics*, vol. 2, no. 1-2, pp. 83–97, March 1955.

[4] R. Burkard, M. Dell'Amico, and S. Martello, *Assignment Problems*. Society for Industrial and Applied Mathematics, 2009.

[5] D. P. Bertsekas, "The auction algorithm: A distributed relaxation method for the assignment problem," *Annals of Operations Research*, vol. 14, pp. 105–123, 1988.

[6] M. Alighanbari and J. P. How, "A robust approach to the UAV task assignment problem," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 2, pp. 118–134, January 2008.

[7] L. Liu and D. A. Shell, "Assessing optimal assignment under uncertainty: an interval-based algorithm," *International Journal of Robotics Research*, vol. 30, pp. 936–953, 2011.

[8] S. S. Ponda, L. B. Johnson, and J. P. How, "Distributed chance-constrained task allocation for autonomous multi-agent teams," in *American Control Conference (ACC)*, June 2012.

[9] A. Shapiro, D. Dentcheva, and A. Ruszczynski, *Lectures on Stochastic Programming: Modeling and Theory, Second Edition*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2014.

[10] E. Delage and S. Mannor, "Percentile optimization for markov decision processes with parameter uncertainty," *Operations Research*, vol. 58, no. 1, pp. 203–213, 2010.

[11] H. Ishii, S. Shiode, T. Nishida, and Y. Namasuya, "Stochastic spanning tree problem," *Discrete Applied Mathematics*, vol. 3, no. 4, pp. 263 – 273, 1981.

[12] E. Nikolova, J. A. Kelner, M. Brand, and M. Mitzenmacher, *Stochastic Shortest Paths Via Quasi-convex Maximization*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 552–563.

[13] E. Nikolova, *Approximation Algorithms for Reliable Stochastic Combinatorial Optimization*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 338–351.

[14] M. Lagoudakis, E. Markakis, D. Kempe, P. Keskinocak, A. Kleywegt, S. Koenig, C. Tovey, A. Meyerson, and S. Jain, "Auction-based multi-robot routing," in *Robotics Science and Systems*, 2005.

[15] C. Bererton, G. Gordon, S. Thrun, and P. Khosla, "Auction mechanism design for multi-robot coordination," in *NIPS*, 2003.

[16] H.-L. Choi, L. Brunet, and J. How, "Consensus-based decentralized auctions for robust task allocation," *IEEE Transactions on Robotics*, vol. 25, no. 4, pp. 912–926, 2009.

[17] A. R. Mosteo and L. Montano, "A survey of multi-robot task allocation," Instituto de Investigacin en Ingeniera de Aragn (I3A), Tech. Rep., 2010.

[18] A. Stentz and M. B. Dias, "A free market architecture for coordinating multiple robots," CMU Robotics Institute, Tech. Rep., 1999.

[19] M. B. Dias and A. Stentz, "A free market architecture for distributed control of a multirobot system," in *6th International Conference on Intelligent Autonomous Systems (IAS-6)*, July 2000, pp. 115 – 122.

[20] N. Kalra, D. Ferguson, and A. Stentz, "Hoplites: A market-based framework for planned tight coordination in multirobot teams," in *Proceedings of the 2005 IEEE International Conference on Robotics and Automation*, April 2005, pp. 1170 – 1177.

[21] B. P. Gerkey and M. J. Mataric, "Sold!: Auction methods for multi-robot coordination," *IEEE Transactions on Robotics*, vol. 18, no. 5, pp. 758–768, October 2002.

[22] L. Parker, "Alliance: an architecture for fault tolerant multirobot cooperation," *IEEE Transactions on Robotics and Automation*, vol. 14, no. 2, pp. 220 –240, apr 1998.

[23] S. S. Ponda, L. B. Johnson, and J. P. How, "Risk allocation strategies for distributed chance-constrained task allocation," in *American Control Conference (ACC)*, June 2013.

[24] S. Lim, C. Sommer, E. Nikolova, and D. Rus, "Practical route planning under delay uncertainty: Stochastic shortest path queries," in *Proceedings of Robotics: Science and Systems*, Sydney, Australia, July 2012.

[25] T. Sargent, *Macroeconomic Theory*. Academic Press, 1979.

[26] R. Horst, *Introduction to global optimization*, ser. Nonconvex optimization and its applications. Dordrecht, Boston: Kluwer Academic Publishers, 2000.